

Exact Results for the BTZ Black Hole

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Abstract

In this review, we summarize exact results for the three-dimensional BTZ black hole. We use rigorous mathematical results to clarify the general structure and properties of this black hole spacetime and its microscopic description. In particular, we study the formation of the black hole by point particle collisions, leading to an exact analytic determination of the Choptuik scaling parameter. We also show that a ‘No Hair Theorem’ follows immediately from a mathematical theorem of hyperbolic geometry, due to Sullivan. A microscopic understanding of the Bekenstein-Hawking entropy, and decay rate for massless scalars, is shown to follow from standard results of conformal field theory.

February 2001

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1 Introduction

The challenge of understanding the microscopic properties of black hole physics is an important step in the quest for a consistent quantum theory of gravity. Recent developments in string theory have led to significant advances in our understanding of the microscopic structure of black holes. The microscopic derivation of the Bekenstein-Hawking entropy of a string theory black hole is perhaps the most notable [1]. This result led to a more thorough investigation of the nature of black holes within the context of string theory; for recent reviews see, for example, [2]-[6]. It became clear that many aspects of a large class of extremal and near-extremal black holes can be given a conformal field theory description. The black holes in question are solutions of the low-energy equations of motion of superstring theory. One of the most significant aspect of these black holes is the fact that they are charged with respect to certain RR-fields. These RR-charges are in turn carried by D-branes in string theory [7]. Exact agreement was found between the Bekenstein-Hawking entropy of a classical five-dimensional extremal RR-charged black hole and the entropy of states of a corresponding D-brane system with the same quantum numbers [1].

On the other hand it is instructive to have at one's disposal a lower-dimensional toy model, as this often affords the possibility of a more detailed analysis. The construction of a black hole spacetime in $(2 + 1)$ dimensions, with negative cosmological constant, provides one such example. This is known as the Bañados-Teitelboim-Zanelli (BTZ) black hole [8], and it warrants study in its own right; for a review, see [9]. A key feature of this model lies in the simplicity of its construction; it is a constant negatively curved spacetime, and it is obtained as a discrete quotient of three-dimensional anti-de Sitter space [10]. In particular, there is no curvature singularity. Even so, all characteristic features such as the event horizon and Hawking radiation are present so that this model is a genuine black hole. Furthermore, despite its simplicity, the BTZ black hole plays a significant role in many of the recent developments in string theory [5], [11]-[13]. In particular, a generic feature of all string theory black holes for which an exact counting of microstates is possible, is that the near horizon geometry of these solutions is that of a BTZ black hole. Our aim in this review is to establish a number of exact results for the kinematical and dynamical properties of the BTZ black hole. The approach presented here is complementary to the semi-classical quantization which focuses on quantum field theory in a black hole background. Here, we are interested in the microscopic properties of the black hole itself. We obtain these results without referring to any specific underlying microscopic theory for gravity such as string theory. While these results shed light on the general structure of the $(2 + 1)$ -dimensional spacetime itself, they also help to understand universal properties of certain higher-dimensional string theory black holes. For a discussion of string theory black holes and their connection to the BTZ-black hole see for example, [4]-[6].

Firstly, we focus on the formation of the black hole from point particle collisions [14]-[16]. We show that formation can be understood at a purely algebraic level, in terms of isometries of anti-de Sitter space. This leads to an exact analytic understanding of the associated Choptuik scaling parameter, and represents the first such exact determination [16]. Interest in black hole formation within the context of numerical relativity has increased due to the critical scaling behaviour discovered by Choptuik [17]. It was observed numerically that the threshold for black hole formation has a simple structure in the space of initial data. In particular, the black hole mass parameter, for example, exhibits a certain universal power-law scaling behaviour; for a review, see [18]. It is important to have a model where this behaviour can be studied exactly

and analytically, and the BTZ black hole provides such an example [16].

Following on from this, we invoke a precise mathematical theorem of hyperbolic geometry, due to Sullivan [19, 20], to establish a ‘No Hair Theorem’ for the BTZ black hole [21]. This result shows that the BTZ black hole can be parametrized by at most two parameters, its mass and angular momentum. Furthermore, the theorem of Sullivan provides a precise notion of holography, whereby the three-dimensional spacetime structure is completely determined in terms of certain two-dimensional boundary data. The relevance of holography in gravitational systems has often been discussed [22, 23]. Again, this concept has received exciting impetus from recent developments in string theory. In particular, the AdS/CFT conjecture of Maldacena [24]–[27] states that string theory defined on an anti-de Sitter background is dual, in a particular limit, to superconformal field theory defined on the boundary of anti-de Sitter space; for a review, see [28]. This holographic conjecture has received enormous attention recently, and has led to important advances in our understanding of the dynamics of gauge theory and gravity. However, while the dynamical conjecture of Maldacena remains to be proved, it is satisfying to have an exact mathematical theorem which establishes a precise notion of holography in a kinematical sense. The fact that the BTZ black hole satisfies the requirements of this theorem again shows the relevance of this model within the wider context of string theory.

As mentioned above, the microscopic derivation of the entropy of string theory black holes was carried out by counting the excitations of certain D-brane configurations at weak coupling where spacetime is flat. Supersymmetry then relates this configuration to the corresponding black hole spacetime with the given RR-charges. An interesting question is whether it is possible to count the microstates without recourse to a specific microscopic realisation in terms of D-branes. It turns out that this is indeed possible due to an important result concerning the asymptotic symmetry algebra of the BTZ black hole. It was shown by Brown and Henneaux [29] that the asymptotic symmetric algebra consists of a Virasoro algebra with both left-moving and right-moving sectors. A formula due to Cardy [30] for the degeneracy of states in a conformal field theory can then be used to give a universal microscopic derivation of the BTZ entropy [13, 31]. Furthermore, this understanding of BTZ entropy lies at the heart of many of the derivations for higher-dimensional string theory black holes [12, 13], [32]–[34]. This is due to the fact that these black holes have a near-horizon geometry containing the BTZ black hole. We also show that an exact convergent expansion for the degeneracy of states of a conformal field theory can be used to compute the corrections to the Bekenstein-Hawking entropy [35]. This expansion, due to Rademacher [36, 37], is a remarkable result from analytic number theory and generalizes the Cardy formula mentioned above. Due to its exact nature, the expansion gives a precise determination of the form of the corrections to the Bekenstein-Hawking entropy formula [35, 38]. An alternative way of counting the BTZ microstates was previously suggested by Carlip [39].

Having discussed the formation process, and resulting kinematical properties, it remains to understand the decay of non-extremal black holes. This involves studying the absorption of quanta by the black hole, and then allowing it to evaporate, via Hawking radiation, back to extremality. In [40]–[45], the low energy scattering cross sections and decay rates for a massless minimally coupled scalar field were computed for a large class of four- and five-dimensional black holes, and agreement was found with conformal field theory or effective string theory predictions. In each of these cases, the result relied on a particular matching of solutions between a near-horizon region and an asymptotic region. For certain ranges of parameters inherent to the problem, this matching agrees with a conformal field theory description.

Here, we study the propagation a massless minimally coupled scalar field in the background geometry of the BTZ black hole. The special feature is that the wave equation can be solved exactly, without any approximations [46, 47]. This allows us to determine exactly the range of energy and angular momentum of the scattered field, for which the the decay rate agrees with a conformal field theory description [48]. We find agreement for energies small in comparison to the size of the black hole, and to the curvature scale of the spacetime; in addition, one is restricted to the zero angular momentum wave. In this region, however, agreement is found for all values of mass and angular momentum, and thus the conformal field theory description is not restricted to a near-extremal limit. Finally, we provide a microscopic derivation of the Hawking decay rate [49]. This is achieved by first showing how the conformal field theory which represents the asymptotic isometries of the BTZ black hole is perturbed by the presence of matter fields. The transition probabilities in the black hole which are induced by this perturbation are then calculated, and we find agreement with the semiclassical Hawking decay rate.

The plan of this article is as follows. In section 2, we present the basic construction and properties of the BTZ black hole. In section 3, we discuss the connection between the Gott time machine and the formation of the BTZ black hole from point particle collisions. This yields an exact analytic determination of the Choptuik scaling parameter. In section 4, we show how Sullivan’s theorem of hyperbolic geometry provides a ‘No Hair Theorem’ for the BTZ black hole, and we also establish the holographic nature of the spacetime. This is followed by the microscopic derivation of the entropy from the Brown-Henneaux algebra. We also use a convergent Rademacher expansion to compute corrections to the Bekenstein-Hawking entropy in an exact way. In section 5, we study the low energy dynamics of the BTZ black hole. In particular, we compute the exact decay rate for massless minimally coupled scalars, and show how it can be interpreted in terms of a left-right symmetric conformal field theory. We present our conclusions in section 6.

2 Construction and Properties

We begin by recalling the metric for the BTZ black hole. In terms of Schwarzschild coordinates, the line element is given by [8, 9]

$$ds^2 = - \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} \right) dt^2 + \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} \right)^{-1} dr^2 + r^2 \left(d\phi - \frac{J}{2r^2} dt \right)^2, \quad (1)$$

where M and J are the mass and angular momentum parameters of the black hole. This metric satisfies the vacuum Einstein equations in 2+1 dimensions, with a negative cosmological constant $\Lambda = -1/l^2$, namely

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{l^2} g_{\mu\nu}. \quad (2)$$

The metric is singular at the location of the inner and outer horizons $r = r_{\pm}$, defined by

$$r_{\pm}^2 = \frac{Ml^2}{2} \left(1 \pm \sqrt{1 - \frac{J^2}{M^2 l^2}} \right). \quad (3)$$

Thus, we can express the mass and angular momentum in terms of r_{\pm} as

$$M = \frac{r_+^2 + r_-^2}{l^2}, \quad J = \frac{2r_+r_-}{l}. \quad (4)$$

We shall use units for which Newton's constant satisfies $8G = 1$. The $M = -1, J = 0$ metric is then recognized as that of three-dimensional anti-de Sitter space (AdS_3). It is separated by a mass gap from the $M = 0, J = 0$ "massless black hole."

For later use, let us recall some elements of the canonical approach to the BTZ black hole [10]. For this, we consider a general stationary, axially symmetric, configuration parametrized by

$$ds^2 = -(N^\perp(r))^2 dt^2 + \frac{dr^2}{f^2(r)} + r^2 (N^\phi(r) dt + d\phi)^2, \quad \phi \in [0, 2\pi), \quad (5)$$

where N^\perp and N^ϕ are the usual lapse and shift functions. Upon solving the constraint equations, one finds that

$$\begin{aligned} N^\perp(r) &= f(r)N(\infty), \\ N^\phi(r) &= -\frac{J}{2r^2}N(\infty) + N^\phi(\infty), \\ f^2(r) &= \frac{r^2}{l^2} - M + \frac{J^2}{4r^2}. \end{aligned} \quad (6)$$

Thus far, $N(\infty), N^\phi(\infty), M$ and J are merely integration constants. On the other hand, if we now consider arbitrary variations $\delta g_{\mu\nu}$ approaching the form (5) for large r , it is easy to see that under such variations the action picks up a boundary contribution

$$\delta S = (t_2 - t_1) [N(\infty)\delta M - N^\phi(\infty)\delta J] + \delta \mathcal{B}, \quad (7)$$

where \mathcal{B} parametrizes the possibility of adding boundary terms to the Einstein-Hilbert action. In order for the variation of the action to vanish for the BTZ solution, we must choose

$$\mathcal{B} = (t_2 - t_1) [-N(\infty)M + N^\phi(\infty)J]. \quad (8)$$

We now see that M and J appear as variables conjugate to the lapse and shift functions, justifying their interpretation as mass and angular momentum, respectively.

It is straightforward to check that any solution of the vacuum Einstein equations (2) is also a space of constant negative curvature. Thus, the BTZ black hole is locally isometric to AdS_3 . Moreover, it is known that the BTZ black hole is obtained by performing a quotient of AdS_3 by a discrete finitely generated group of isometries of AdS_3 . For completeness, we include the relevant parts of this construction, as it plays a crucial role in the remaining sections.

A standard representation [9] of AdS_3 may be obtained from a flat spacetime $\mathbf{R}^{2,2}$, with coordinates (X_1, X_2, T_1, T_2) , and metric

$$ds^2 = dX_1^2 + dX_2^2 - dT_1^2 - dT_2^2. \quad (9)$$

The induced metric on the submanifold

$$X_1^2 + X_2^2 - T_1^2 - T_2^2 = -l^2, \quad (10)$$

then corresponds to the AdS_3 metric. From (10), it is clear that the isometry group of AdS_3 is $SO(2, 2)$. Equivalently, we can combine the coordinates into an $SL(2, \mathbf{R})$ matrix

$$X = \frac{1}{l} \begin{pmatrix} T_1 + X_1 & T_2 + X_2 \\ -T_2 + X_2 & T_1 - X_1 \end{pmatrix}, \quad (11)$$

with $\det X = 1$. Hence, AdS_3 can be viewed as the group manifold of $SL(2, \mathbf{R})$, with the isometries being represented by left and right multiplication via $X \rightarrow \rho_L X \rho_R$. Here, ρ_L and ρ_R are elements of $SL(2, \mathbf{R})$. One has a further \mathbf{Z}_2 identification of isometries, namely $(\rho_L, \rho_R) \sim (-\rho_L, -\rho_R)$. This follows from the fact that $SO(2, 2) \approx SL(2, \mathbf{R}) \times SL(2, \mathbf{R})/\mathbf{Z}_2$.

The key point in the construction of the BTZ black hole is to recognize that the requirement of periodicity in the angular coordinate ϕ is implemented by identifying points of anti-de Sitter space by the isometry (ρ_L, ρ_R) , where

$$\rho_L = \begin{pmatrix} e^{\pi(r_+ - r_-)/l} & 0 \\ 0 & e^{-\pi(r_+ - r_-)/l} \end{pmatrix}, \quad \rho_R = \begin{pmatrix} e^{\pi(r_+ + r_-)/l} & 0 \\ 0 & e^{-\pi(r_+ + r_-)/l} \end{pmatrix}. \quad (12)$$

The BTZ black hole is then given [9, 10] as the quotient space $AdS_3/\langle(\rho_L, \rho_R)\rangle$, where $\langle(\rho_L, \rho_R)\rangle$ denotes the group generated by the isometry (ρ_L, ρ_R) .

We shall also require certain properties of the Euclidean section of the black hole [9, 50]. The metric is obtained from (1) by performing the continuation $t = i\tau_E$ and $J = -iJ_E$, leading to the line element

$$ds_E^2 = \left(-M + \frac{r^2}{l^2} - \frac{J_E^2}{4r^2}\right) d\tau_E^2 + \left(-M + \frac{r^2}{l^2} - \frac{J_E^2}{4r^2}\right)^{-1} dr^2 + r^2 \left(d\phi - \frac{J_E}{2r^2} d\tau_E\right)^2. \quad (13)$$

The parameters r_{\pm} are now given by

$$r_+ = \left[\frac{Ml^2}{2} \left(1 + \sqrt{1 + \frac{J_E^2}{M^2 l^2}} \right) \right]^{1/2}, \quad r_- = \left[\frac{Ml^2}{2} \left(1 - \sqrt{1 + \frac{J_E^2}{M^2 l^2}} \right) \right]^{1/2} \equiv -i |r_-|. \quad (14)$$

Three-dimensional hyperbolic space (a space of constant negative curvature with Euclidean signature) is denoted by H^3 . The Euclidean BTZ black hole is obtained as a quotient of H^3 by a discrete finitely generated group of isometries of H^3 . The precise form of the identifications can be seen by writing the metric for H^3 in the upper-half space coordinates (x, y, z) defined by [9, 50]

$$\begin{aligned} x &= \left(\frac{r^2 - r_+^2}{r^2 - r_-^2} \right)^{1/2} \cos \left(\frac{r_+}{l^2} \tau_E + \frac{|r_-|}{l} \phi \right) \exp \left(\frac{r_+}{l} \phi - \frac{|r_-|}{l^2} \tau_E \right), \\ y &= \left(\frac{r^2 - r_+^2}{r^2 - r_-^2} \right)^{1/2} \sin \left(\frac{r_+}{l^2} \tau_E + \frac{|r_-|}{l} \phi \right) \exp \left(\frac{r_+}{l} \phi - \frac{|r_-|}{l^2} \tau_E \right), \\ z &= \left(\frac{r_+^2 - r_-^2}{r^2 - r_-^2} \right)^{1/2} \exp \left(\frac{r_+}{l} \phi - \frac{|r_-|}{l^2} \tau_E \right). \end{aligned} \quad (15)$$

The metric becomes

$$ds_E^2 = \frac{l^2}{z^2} (dx^2 + dy^2 + dz^2), \quad z > 0. \quad (16)$$

It follows that the BTZ metric is locally isometric to H^3 . However, from (15), we see that periodicity in the Schwarzschild coordinate ϕ is implemented [9, 50] via the identifications

$$(x, y, z) \sim e^{2\pi r_+/l} \left(x \cos \left(\frac{2\pi |r_-|}{l} \right) - y \sin \left(\frac{2\pi |r_-|}{l} \right), x \sin \left(\frac{2\pi |r_-|}{l} \right) + y \cos \left(\frac{2\pi |r_-|}{l} \right), z \right). \quad (17)$$

To determine the topology of the resulting Euclidean black hole, we introduce spherical coordinates on the upper-half space defined by [9, 50]

$$(x, y, z) = (R \cos \theta \cos \chi, R \sin \theta \cos \chi, R \sin \chi), \quad (18)$$

with $\theta \in [0, 2\pi]$, $\chi \in [0, \pi/2]$. The line element is then written as

$$ds_E^2 = \frac{l^2}{\sin^2 \chi} \left(\frac{dR^2}{R^2} + d\chi^2 + \cos^2 \chi d\theta^2 \right), \quad (19)$$

and the identifications (17) become

$$(R, \theta, \chi) \sim \left(R e^{2\pi r_+/l}, \theta + \frac{2\pi |r_-|}{l}, \chi \right). \quad (20)$$

A fundamental region is the space between the hemispheres $R = 1$ and $R = e^{2\pi r_+/l}$ with the inner and outer boundaries identified along a radial line, followed by a $2\pi r_+/l$ rotation about the z -axis. Topologically, the resulting manifold is a solid torus [9, 50]. For $\chi \neq \pi/2$, each slice of fixed χ is a 2-torus with periodic coordinates $\ln R$ and θ .

Finally, we recall that the Hawking temperature T_H , the length of the event horizon A_H , and the angular velocity at the event horizon Ω_H , are given by [9]

$$T_H = \frac{r_+^2 - r_-^2}{2\pi \ell^2 r_+}, \quad A_H = 2\pi r_+, \quad \Omega_H = \frac{J}{2r_+^2}. \quad (21)$$

3 Formation and Choptuik Scaling

Within the context of numerical relativity, one of the most significant recent results is the evidence for Choptuik scaling in the formation of black hole spacetimes [17]. In particular, the threshold for black hole formation in the space of initial data was observed to have a surprisingly simple structure. Specifically, one considers a generic smooth one-parameter family of initial data (labelled by p), such that for large values of the parameter p a black hole is formed, while no black hole is formed for small p . The mass M of the black hole then satisfies the Choptuik scaling relation

$$M \simeq C(p - p_*)^\gamma, \quad (22)$$

in the limit $p \sim p_*$ with $p > p_*$. The constant γ is the same for all such one-parameter families, and takes the numerical value $\gamma \sim 0.37$, for a four-dimensional black hole. The parameter γ is known as the Choptuik scaling parameter. Here, C depends on the initial data.

It is of interest to have a model of black hole formation in which the Choptuik scaling can be calculated in an exact and analytic fashion. Our aim here is to show that indeed this can be achieved for the BTZ black hole. The key observation is to establish a connection between two seemingly different spacetime constructions, the Gott time machine on the one hand, and the BTZ black hole on the other [16]. We show how the construction of the Gott time machine leads to a precise formulation of the order parameter for BTZ black hole formation. An exact determination of the corresponding Choptuik scaling follows immediately.

3.1 The Gott Time Machine

In [51], a precise mechanism was presented for the production of closed timelike curves. In particular, the spacetime of two point particles with mass and boost parameters α and ξ , in $(2+1)$ -dimensional spacetime with vanishing cosmological constant Λ , was shown to produce closed timelike curves if the inequality, $\sin \frac{\alpha}{2} \cosh \xi > 1$, is satisfied.

In [52, 53], this Gott time machine was analysed in terms of the group theoretic approach to point particles [54]. We first note that the $SO(2,1)$ Lorentz group of Minkowski space is locally equivalent to $SL(2, \mathbf{R})$. The essential point to recall is that elements of $SL(2, \mathbf{R})$ are classified according to the value of their trace. We have

$$\begin{aligned} |\text{Tr } T| &< 2, & \text{Elliptic (Timelike)}, \\ |\text{Tr } T| &= 2, & \text{Parabolic (Lightlike)}, \\ |\text{Tr } T| &> 2, & \text{Hyperbolic (Spacelike)}. \end{aligned} \tag{23}$$

In the following, we shall use the equivalent $SU(1,1)$ notation instead of $SL(2, \mathbf{R})$; they are locally equivalent and related by conjugation, which is given explicitly in [55]. As shown in [54], the spacetime for a single static point particle with $\Lambda = 0$ is obtained by removing a wedge of deficit angle α , and identifying opposite sides of the wedge. The particle spacetime is defined via the rotation generator with angle α ,

$$R(\alpha) = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}. \tag{24}$$

The mass m of the particle is given by $\alpha = \pi m$, in units with $8G = 1$, and the resulting spacetime has a naked conical singularity.

A moving particle is obtained by boosting to the rest frame of the particle, rotating, and then boosting back. Thus, the generator for a moving particle is

$$T = B(\boldsymbol{\xi})R(\alpha)B^{-1}(\boldsymbol{\xi}), \tag{25}$$

where the boost matrix is given by

$$B(\boldsymbol{\xi}) = \begin{pmatrix} \cosh \frac{\xi}{2} & e^{-i\phi} \sinh \frac{\xi}{2} \\ e^{i\phi} \sinh \frac{\xi}{2} & \cosh \frac{\xi}{2} \end{pmatrix}. \tag{26}$$

Here, $\boldsymbol{\xi}$ is the boost vector with $\xi = |\boldsymbol{\xi}|$, and ϕ is the polar angle.

To construct the Gott time machine, we consider a two-body collision process, with particles labelled by A and B . The effective two-particle generator is then the product $T^G = T_B T_A$ [52]-[54]. The central object of interest to us is the trace of this generator. It is straightforward to compute

$$\begin{aligned} \frac{1}{2} \text{Tr } T^G &= \cos \frac{\alpha_A}{2} \cos \frac{\alpha_B}{2} + \sin \frac{\alpha_A}{2} \sin \frac{\alpha_B}{2} \\ &- \sin \frac{\alpha_A}{2} \sin \frac{\alpha_B}{2} \left[\cosh^2 \left(\frac{\xi_A + \xi_B}{2} \right) + \cosh^2 \left(\frac{\xi_A - \xi_B}{2} \right) \right] \\ &+ \sin \frac{\alpha_A}{2} \sin \frac{\alpha_B}{2} \cos(\phi_A - \phi_B) \left[\cosh^2 \left(\frac{\xi_A + \xi_B}{2} \right) - \cosh^2 \left(\frac{\xi_A - \xi_B}{2} \right) \right]. \end{aligned} \quad (27)$$

The original Gott time machine is recovered by choosing particles with equal masses, and equal and opposite boosts, namely $\alpha_A = \alpha_B = \alpha$, $\xi_A = \xi_B = \xi$, $\phi_A - \phi_B = \pi$. We find

$$\frac{1}{2} \text{Tr } T^G = 1 - 2 \sin^2 \frac{\alpha}{2} \cosh^2 \xi. \quad (28)$$

When the Gott condition is satisfied, we have $\sin^2 \frac{\alpha}{2} \cosh^2 \xi > 1$, and thus T^G is a hyperbolic generator, and when $\sin^2 \frac{\alpha}{2} \cosh^2 \xi < 1$, we have an elliptic generator. Thus, the effective two-particle generator (the Gott time machine) becomes hyperbolic (spacelike) precisely when the Gott condition is satisfied. The physical consistency of this spacetime has been discussed in [52, 53, 56, 57].

3.2 Choptuik Scaling

We are interested in applying this algebraic construction to study the formation of the BTZ black hole. For negative cosmological constant, the static and moving particle spacetimes are defined in a fashion analogous to the above, except that one has both left and right generators. Particle spacetimes for non-zero cosmological constant have been constructed in [58]. For our purposes here, the most important aspect of the BTZ black hole is that it is defined by a hyperbolic isometry. One may choose a fundamental region of this hyperbolic isometry, and define the black hole spacetime by identification of the region's boundaries by the isometry.

We recall that the conventional mass parameter of the BTZ black hole is denoted by M , while the point particle mass m is related by $m = 2(1 - \sqrt{-M})$. As a result, the point particle mass spectrum is $-1 < M < 0$, while the black hole mass spectrum is $M \geq 0$, with $M = -1$ corresponding to AdS_3 .

Let us consider the static black hole case, in which the left and right generators are taken to be equal [9]. Since the isometries of AdS_3 are subject to the identification $(\rho_L, \rho_R) \sim (-\rho_L, -\rho_R)$, we may take $\rho_L = \rho_R = -T^G \equiv \rho$. If the Gott condition is satisfied, then as we have seen T^G is a hyperbolic generator, and consequently the Gott time machine results in BTZ black hole formation. The black hole mass is then given by [9]

$$\frac{1}{2} \text{Tr } \rho = \cosh \pi \sqrt{M} = -1 + 2 \sin^2 \frac{\alpha}{2} \cosh^2 \xi \equiv p, \quad (29)$$

where $p \geq 1$.

It is important to consider the defining equation of this process, namely

$$T_B T_A = T^G. \quad (30)$$

On the left-hand side, we have the input data given by the particle mass and boost parameters α and ξ . The incoming particles A and B have been set up in a symmetrical way, with equal mass and boost parameters, and the timelike geodesics representing their worldlines will intersect at a given time, say $t = 0$. Thus, we may regard this as the time of collision of the two particles. The product of the particle generators represents the effective generator of the system at this time [14]. As we have seen, the effective generator at the time of collision is hyperbolic if the Gott condition is satisfied. We may then interpret equation (30) as defining the formation of a BTZ black hole at time $t = 0$, with the value of the black hole mass fixed by the input parameters α and ξ . Thus, equation (30) encodes dynamical information. However, the precise details of the motion of the particles corresponding to the generators T_A and T_B prior to the collision, as well as the motion after collision may also be studied. Indeed, this analysis has been performed for massless particles in [14, 15].

We see from (29) that the natural order parameter for black hole formation in $(2 + 1)$ -dimensional anti-de Sitter gravity is the trace of the generator. This takes a critical value at the threshold for black hole formation, corresponding to the critical value of the parameter $p_* = 1$. Clearly, $p = p_*$ corresponds to the black hole vacuum $M = 0$, where the Gott generator is parabolic. Since the parameter p depends on the initial data α and ξ , we can read off the critical boost ξ for any given mass α . We have

$$\pi\sqrt{M} = \operatorname{arccosh} p = \ln \left[p + \sqrt{p^2 - 1} \right]. \quad (31)$$

We stress that the above expression is an exact analytic formula for the formation of a BTZ black hole in terms of the input (initial) parameters α and ξ , equivalently p . From this, we can immediately determine the Choptuik scaling by studying the behaviour near p_* . The mass M and horizon length r_+ are related by $\sqrt{M} = r_+/l$. Writing $p = p_* + \epsilon$, we find to leading order

$$\frac{r_+}{l} = \frac{\sqrt{2}}{\pi} (p - p_*)^{1/2}. \quad (32)$$

Thus, we observe that the horizon length scales with a factor of $1/2$. Note that this is indeed a universal scaling since BTZ black hole formation always requires a hyperbolic generator. The universal scaling value of $1/2$ is simply a consequence of the fact that the horizon length depends on the inverse cosh function. One can equally well express the scaling behaviour directly in terms of the mass, as in (22).

If the Gott condition is not satisfied, then one has an effective particle spacetime with an elliptic generator, whose effective deficit angle is denoted by α_{eff} . This can be obtained by continuation of (29) to negative M values. We find,

$$\frac{1}{2} \operatorname{Tr} \rho = \cos \pi \sqrt{-M} = -1 + 2 \sin^2 \frac{\alpha}{2} \cosh^2 \xi \equiv p, \quad (33)$$

where now $p < p_*$. Once again, we have an exact analytic expression for the mass parameter on the other side of the transition, and of course the Choptuik scaling exponent is again $\gamma = 1/2$, with

$$\alpha_{\text{eff}} = 2\pi - 2\sqrt{2}(p_* - p)^{1/2}. \quad (34)$$

One can also consider Choptuik scaling for the spinning black hole, and in this case we need independent left and right generators. One finds that $(r_+ \pm r_-)$ both have a scaling value of $1/2$ [16]. The problems of closed timelike curves and chronology protection [59] are overcome by the creation of the black hole horizon, as soon as the Gott condition is satisfied. We mention that the recent study of BTZ black hole formation from massless particle collisions [14, 15] is based on the lightlike analogue of the Gott time machine [60], and thus is also guaranteed to produce the hyperbolic generator necessary for BTZ formation. The holographic description of the creation process has been investigated within the AdS/CFT correspondence in [61]-[64]. The formation of the BTZ black hole from gravitational collapse was first considered in [65], and a Choptuik scaling value of $1/2$ was also observed for collapsing dust shells in [66]. Furthermore, a precise interpretation of this Choptuik scaling parameter in terms of the timescale for return to equilibrium of the boundary conformal field theory has recently been given [67]. Numerical studies in $(2+1)$ dimensions have also been performed recently with different scaling values. This is explained by the nature of the transition to the black hole phase [68]-[71].

4 Kinematical Properties: Holography and Entropy

The formulation of a precise holographic principle for gravitating systems has been widely discussed. Within the context of string theory, the recent conjecture due to Maldacena provides a concrete testing ground for such ideas. In essence, the Maldacena conjecture establishes a duality between string theory defined on an anti-de Sitter background, and a corresponding superconformal field theory defined on the boundary of anti-de Sitter space. In $2+1$ dimensions, the Maldacena conjecture can be investigated more thoroughly, since details are known about both sides of the AdS/CFT correspondence [37, 72]. Since solutions to Einstein's equations with negative cosmological constant form the basic ingredients in the correspondence, it is clear that the BTZ black hole will play an important role in verifying the Maldacena conjecture.

In this section, we will demonstrate that a precise mathematical notion of holography can be defined in the context of three-dimensional hyperbolic geometry. In particular, we invoke a theorem of Sullivan to establish the exact sense in which the BTZ black hole is a holographic manifold [21]. This theorem may also be interpreted as a ‘No Hair Theorem.’

We then show how the asymptotic symmetry algebra of $(2+1)$ -dimensional gravity (Brown-Henneaux algebra) can be used to give a microscopic derivation of the Bekenstein-Hawking entropy for non-extremal BTZ black holes [13, 31]. Furthermore, we employ an exact result from analytic number theory (a Rademacher expansion) to compute precisely the corrections to the Bekenstein-Hawking entropy [35].

4.1 Sullivan's Theorem

The theorem of Sullivan [19, 20] provides a precise mathematical statement of a holographic principle, i.e., a three-dimensional structure being completely determined in terms of a two-dimensional structure. Essentially, the theorem states that the inequivalent hyperbolic structures of a three-dimensional geometrically finite Kleinian manifold (a term to be defined) are parametrized by the Teichmüller space of the boundary. In this section, we explore the consequences of this theorem in the context of the Euclidean BTZ black hole.

Let us consider the upper half-space model for hyperbolic 3-space [73], with metric given by (16). The boundary at infinity is the $z = 0$ plane and the point $z = \infty$, namely a 2-sphere S_∞^2 . Let Γ be a discrete subgroup of isometries of H^3 . We denote by Γ_x the orbit of any point of H^3 under the action of Γ . The *limit set* of Γ is defined as

$$L_\Gamma = \overline{\Gamma}_x \cap S_\infty^2. \quad (35)$$

Thus, the limit set is the intersection of the closure of Γ_x with the sphere at infinity, for any point $x \in H^3$. One can show that the limit set is independent of the choice of x , as we will demonstrate explicitly in the case at hand. Given the limit set L_Γ , one now defines the *convex hull* $H(L_\Gamma)$ to be the smallest convex set in H^3 containing L_Γ . We recall that a convex set in H^3 is one which contains all geodesics joining any two points in the set. The associated *convex core* is obtained as a quotient

$$C(\Gamma) = H(L_\Gamma)/\Gamma. \quad (36)$$

Thus, the convex core is the quotient by Γ of the smallest convex set in H^3 containing all geodesics with both end points in the limit set.

We see that the action of Γ partitions S_∞^2 into the limit set L_Γ and its complement Ω , which is called the *domain of discontinuity*. The resulting Kleinian manifold

$$N = (H^3 \cup \Omega)/\Gamma, \quad (37)$$

is then a 3-manifold with a hyperbolic structure on its interior and a complex structure on its boundary [73]. A hyperbolic 3-manifold is said to be a *geometrically finite* Kleinian manifold if the volume of the convex core of Γ is finite. As shown in [73], there are several equivalent definitions of geometrical finiteness. The main theorem may now be stated as follows. Let M denote a topological 3-manifold, and let $GF(M)$ denote the space of geometrically finite hyperbolic 3-manifolds N which are homeomorphic to M . Thus, $GF(M)$ denotes the space of realizations of M by geometrically finite Kleinian manifolds N . Then, we have the following theorem due to Sullivan [19], see also [20].

Theorem: As long as M admits at least one hyperbolic realization, there is a 1-1 correspondence between hyperbolic structures on M and conformal structures on ∂M , i.e.,

$$GF(M) \cong \text{Teich}(\partial M), \quad (38)$$

where $\text{Teich}(\partial M)$ is the Teichmüller space of ∂M .

In order to show that the BTZ black hole is a geometrically finite Kleinian manifold, we must first determine the limit set. If we let γ denote the identification defined by (17), then clearly the BTZ group is a cyclic Kleinian group with elements

$$\Gamma_{\text{BTZ}} = \{\gamma^n, n \in \mathbf{Z}\}. \quad (39)$$

The orbit of any point in H^3 under the action of the BTZ group (17) is easily obtained in the upper-half space model. A simple geometrical picture of the orbit of any point is obtained by noting that the orbit has a helical structure, whereby points are rotated around the z -axis, as they are translated upwards or downwards along the z -axis. Such isometries are referred to as loxodromic, with the z -axis called the loxodromic axis [73]. Thus, the limit set of Γ_{BTZ} consists of two points, corresponding to $n \rightarrow \pm\infty$. Note that the limit set is independent of the chosen point, as required.

Kleinian groups which have a finite limit set are called elementary, and are known to be geometrically finite [74]. Since the limit set of Γ_{BTZ} consists of two points, it follows that Γ_{BTZ} is elementary. Hence, the BTZ black hole is a geometrically finite Kleinian manifold. Furthermore, as shown in section 2, the topology is that of a solid torus.

4.2 Consequences of Sullivan's Theorem

• **No Hair Theorem:** The Euclidean BTZ black hole has the topology of a solid torus. Since the Teichmüller space of the torus is parametrized by two real parameters, the theorem can be interpreted as a ‘No Hair Theorem.’ We conclude that BTZ black hole can be parametrized by at most two parameters (mass and angular momentum), preventing the construction of a charged rotating generalization as a geometrically finite Kleinian manifold with solid torus topology.

• **Holography:** The theorem declares that the BTZ black hole is a holographic manifold, such that the three-dimensional hyperbolic structure is in 1-1 correspondence with the Teichmüller parameters of the two-dimensional genus one boundary.

• **Entropy:** As the Bekenstein-Hawking entropy formula is a geometrical quantity (the length of the horizon), it is determined once the hyperbolic structure is fixed. Hence, the Bekenstein-Hawking entropy is determined by the Teichmüller space on the boundary.

• **Maldacena Conjecture:** In [72], the action for the Euclidean BTZ black hole and H^3 (with periodic time) were written in terms of a complex temperature parameter τ defined on the boundary torus. The resulting form of the action then suggested the existence of an $SL(2, \mathbf{Z})$ family of solutions whose boundary data τ is related by the associated modular transformations. We see that the above theorem does indeed establish the existence of this class of hyperbolic geometries. Furthermore, it also follows that two such geometries whose Teichmüller parameters are related by a modular transformation are equivalent as hyperbolic structures. In [37], the Maldacena conjecture was studied in detail for string theory defined on $AdS_3 \times S^3 \times K3$. The above $SL(2, \mathbf{Z})$ family of solutions was constructed and played an important role in the investigation.

4.3 The Bekenstein-Hawking Entropy

It was observed in [29] that the asymptotic symmetry algebra of $(2+1)$ -dimensional anti-de Sitter gravity realises a left-moving and right-moving Virasoro algebra, with a central charge $c = \bar{c} = 12l$ (in units with $8G = 1$). To see this, we consider the asymptotic form of the BTZ metric (1)

$$ds^2 \rightarrow r^2 dx^+ dx^- + l^2 \frac{dr^2}{r^2} , \quad (40)$$

where $x^\pm = \phi \pm t/l$ are the light-cone coordinates on the AdS -boundary. This asymptotic form is invariant under the infinitesimal transformations $x \rightarrow x + \xi$ with

$$\xi^\pm = f^\pm(x^\pm) , \quad \xi^r = -r \left(f^{+\prime}(x^+) + f^{-\prime}(x^-) \right) . \quad (41)$$

The ξ^\pm generate precisely the group of conformal transformations in $1 + 1$ dimensions. The generators from a representation of the Virasoro algebra

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} , \quad (42)$$

and analogously for the left handed generators \bar{L}_m . The central charge c is representation dependent. The lowest Fourier components L_0 and \bar{L}_0 generate global time translations and rotations. Comparison with (8) then leads to the identification

$$Ml = L_0 + \bar{L}_0, \quad J = L_0 - \bar{L}_0 . \quad (43)$$

Furthermore, it can be shown [29] that value of the central charge c is a consequence of the “improvement term” ξ^r , in much the same way as it appears in constrained WZW models, for example. The physical interpretation of this result is that, whatever the microscopic theory of the BTZ black hole, it will have to form a representation of the conformal algebra (42).

To understand the connection of the algebra (42) with the black hole entropy, we use a result due to Cardy [30]. It relates the asymptotic degeneracy of states $d(N, \bar{N})$ for fixed $L_0 = N$ and $\bar{L}_0 = \bar{N}$ to the central charge of a unitary, modular invariant conformal field theory by

$$d(N, \bar{N}) = \exp \left[2\pi \sqrt{\frac{c}{6}N} + 2\pi \sqrt{\frac{c}{6}\bar{N}} \right] . \quad (44)$$

Let us first consider an extremal black hole $J = Ml$ ($\bar{L}_0 = 0$). Taking the logarithm of the degeneracy of states and using (44), (43), and (3), we find the microcanonical entropy

$$S = 2\pi \sqrt{\frac{c}{6}L_0} = 4\pi r_+ . \quad (45)$$

We thus have agreement between the entropy of the asymptotic conformal field theory and the classical Bekenstein-Hawking formula. Since the Cardy formula is valid for large eigenvalues of L_0 and \bar{L}_0 , this derivation holds for very massive black holes.

Furthermore, it was observed by Strominger [13] that in fact the Cardy formula can be used to give a microscopic derivation of the non-extremal BTZ black hole, without recourse to string theory or supersymmetry. The Cardy formula takes the general form

$$S = 2\pi \sqrt{\frac{c}{6}L_0} + 2\pi \sqrt{\frac{c}{6}\bar{L}_0} . \quad (46)$$

Using (43), one then finds that

$$S = 4\pi r_+ . \quad (47)$$

However, in order for this scenario to be acceptable, we must have a consistent theory of quantum gravity on AdS_3 . One possible realization of such a consistent quantum theory is to embed the BTZ black hole as a solution of string theory [11, 31]. The significance of the BTZ black hole entropy for previous D-brane computations was also realized [12, 13]. In particular, it was noticed that the five-dimensional black hole studied in [1] has a near-horizon geometry of the form $AdS_3 \times S^2$. As a result, one could give a derivation of the entropy based solely on the Brown-Henneaux algebra, with a central charge defined in terms of the D-brane charges.

4.4 The Rademacher Expansion

We have seen how the microscopic degrees of freedom of a conformal field theory encode information about the entropy of a macroscopic black hole, via the Cardy formula. In deriving this formula, the starting point is to consider the modular invariant partition function of a unitary conformal field theory defined on a two-torus, namely

$$Z(\tau) = \text{Tr } e^{2\pi i(L_0 - \frac{c}{24})\tau}. \quad (48)$$

Here, τ is the modular parameter and c is the central charge of the conformal field theory. The Fourier expansion of the partition function takes the form

$$Z(\tau) = \sum_{n \geq 0} F(n) e^{2\pi i(n - \frac{c}{24})\tau}. \quad (49)$$

The black hole entropy in this framework is given by $S = \ln F(n)$, for large n . To study S , we use an exact convergent expansion, due to Rademacher [36], for the Fourier coefficients of a modular form of weight ω . This is given by [37]

$$\begin{aligned} F(n) &= 2\pi \sum_{m - \frac{c}{24} < 0} \left(\frac{n - \frac{c}{24}}{|m - \frac{c}{24}|} \right)^{(\omega-1)/2} F(m) \cdot \\ &\cdot \sum_{k=1}^{\infty} \frac{1}{k} Kl \left(n - \frac{c}{24}, m - \frac{c}{24}; k \right) I_{1-\omega} \left(\frac{4\pi}{k} \sqrt{|m - \frac{c}{24}|(n - \frac{c}{24})} \right). \end{aligned} \quad (50)$$

Here, $I_{1-\omega}$ is the standard Bessel function and $Kl(n, m; k)$ is a Kloosterman sum defined by [37],

$$Kl(n, m; k) = \sum_{d \in (\mathbf{Z}/k\mathbf{Z})^*} \exp \left[\frac{2\pi i}{k} (dn + d^{-1}m) \right]. \quad (51)$$

We are interested in the convergent expansion of $F(n)$ for $\omega = 0$. For large n , this will correspond to the density of states in the conformal field theory for large eigenvalues of L_0 . It is of interest to examine the nature of the leading terms in the Rademacher expansion. These are given by setting $m = 0$ in (50), and furthermore restricting to the $k = 1$ term. We find

$$F(n) = \frac{(2\pi)^{3/2}}{12} c S_0^{-3/2} e^{S_0} \left[1 - \frac{3}{8S_0} - \dots \right], \quad (52)$$

where

$$S_0 = 2\pi \sqrt{\frac{c}{6} \left(n - \frac{c}{24} \right)}, \quad (53)$$

and we have used the asymptotic expansion of the Bessel function. The leading terms in the black hole entropy are then given by [35, 38]

$$S = S_0 - \frac{3}{2} \ln S_0 + \ln c + \text{constant}. \quad (54)$$

This reveals the presence of a logarithmic correction to the Bekenstein-Hawking entropy S_0 . This can also be understood as arising from the power-like factor multiplying the asymptotic density of states [37].

As an example, for the extremal BTZ black hole considered above, one finds a logarithmic correction to the entropy of the form $-3/2 \ln(A/4G)$. In [38], the original derivation of the Cardy formula was extended to include the first subleading correction. Indeed, (54) is in precise agreement with the formula derived in [38]. One interesting point to note is that the $-3/2 \ln S_0$ term first appeared in the quantum geometry formalism [75, 76]. However, the main point to stress here is that the Rademacher expansion is an exact convergent expression which determines all subleading corrections.

5 Dynamical Properties: Decay Rate

We have already seen that conformal field theory plays a crucial role in the kinematical properties of the BTZ black hole. However, the fact that dynamical issues such as the decay rate of non-extremal black holes can also be given a conformal field theory interpretation is perhaps more surprising. As already mentioned, following the successful D-brane derivation of the entropy of extremal black holes in string theory, the properties of non-extremal black holes were studied. Remarkably, it was shown that Hawking radiation of near-extremal black holes also has an analogue in the emission of closed strings in D-brane dynamics; for a review, see, for example, [2, 3, 4, 6]. In the original examples which were studied [40]-[45], the calculations required a certain matching of solutions to the wave equation in an overlap region between the near-horizon and asymptotic regions.

We begin this section by studying the propagation of a massless minimally coupled scalar field in the BTZ background. The key point is that once again this model allows for an exact calculation, without the necessity of any matching procedure [46, 47]. Consequently, we can determine precisely the range of energy and angular momentum of the scattered field, for which the decay rate is consistent with a conformal field theory description [48]. In particular, we find that the latter description is not restricted to the near-extremal limit. The decay rate for various other fields is presented in [77]-[79]. In the second part, we give a microscopic derivation of this semiclassical decay rate by working out the perturbation of the conformal field theory realised by the BTZ black hole.

5.1 The Wave Equation

It is known that the minimally coupled scalar field equation can be solved exactly in the background geometry of the BTZ black hole [46, 47]. This allows an exact determination of the scattering cross section and decay rate of the scalar field. The scalar wave equation $\nabla^2 \Psi = 0$ takes the form

$$\left(-f^{-2} \partial_t^2 + f^2 \partial_r^2 + \frac{1}{r} (\partial_r r f^2) \partial_r - \frac{J}{r^2} f^{-2} \partial_t \partial_\phi - \frac{A}{r^2} f^{-2} \partial_\phi^2 \right) \Psi = 0, \quad (55)$$

where

$$f^2 = \frac{1}{l^2 r^2} (r^2 - r_-^2)(r^2 - r_+^2), \quad A = M - \frac{r^2}{l^2}. \quad (56)$$

Employing the ansatz

$$\Psi(r, t, \phi) = R(r, \omega, m) e^{-i\omega t + im\phi}, \quad (57)$$

leads to the radial equation for $R(r)$

$$\partial_r^2 R(r) + \left(-\frac{1}{r} + \frac{2r}{r^2 - r_-^2} + \frac{2r}{r^2 - r_+^2} \right) \partial_r R(r) + f^{-4} \left(\omega^2 - \frac{J\omega m}{r^2} + \frac{Am^2}{r^2} \right) R(r) = 0. \quad (58)$$

Upon the change of variables $z = \frac{r^2 - r_+^2}{r^2 - r_-^2}$, the radial equation becomes

$$z(1-z)\partial_z^2 R(z) + (1-z)\partial_z R(z) + \left(\frac{A_1}{z} + B_1 \right) R(z) = 0, \quad (59)$$

where

$$A_1 = \left(\frac{\omega - m\Omega_H}{4\pi T_H} \right)^2, \quad B_1 = -\frac{x_-}{x_+} \left(\frac{\omega - m\Omega_H \frac{x_+}{x_-}}{4\pi T_H} \right)^2. \quad (60)$$

The hypergeometric form of (59) becomes explicit upon removing the pole in the last term through the ansatz

$$R(z) = z^\alpha g(z), \quad \alpha^2 = -A_1. \quad (61)$$

We then have

$$z(1-z)\partial_z^2 g(z) + (2\alpha+1)(1-z)\partial_z g(z) + (A_1 + B_1)g(z) = 0. \quad (62)$$

In the neighbourhood of the horizon, $z = 0$, two linearly independent solutions are then given by $F(a, b, c, z)$ and $z^{1-c}F(a-c+1, b-c+1, 2-c, z)$, where

$$\begin{aligned} a+b &= 2\alpha, \\ ab &= \alpha^2 - B_1, \\ c &= 1 + 2\alpha. \end{aligned} \quad (63)$$

Note that $c = a + b + 1$.

5.2 Semiclassical Decay Rate

We choose the solution which has ingoing flux at the horizon, namely,

$$R(z) = z^\alpha F(a, b, c, z). \quad (64)$$

To see this, we note that the conserved flux for (58) is given, up to an irrelevant normalisation, by

$$\mathcal{F} = \frac{2\pi}{i} (R^* \Delta \partial_r R - R \Delta \partial_r R^*), \quad (65)$$

where $\Delta = rf^2$. The flux can be evaluated by noting that

$$\Delta \partial_r = \frac{2\Delta_-}{l^2} z \partial_z, \quad (66)$$

where $\Delta_- = r_+^2 - r_-^2$. Then, using the fact that ab is real, we find the total flux (which is independent of z) to be given by

$$\mathcal{F}(0) = \frac{8\pi\Delta_-}{l^2} \text{Im}[\alpha] |F(a, b, c, 0)|^2 = 2\mathcal{A}_H(\omega - m\Omega_H). \quad (67)$$

In order to compute the absorption cross section, we need to divide (67) by the ingoing flux at infinity. The distinction between ingoing and outgoing waves is complicated by the fact that the BTZ spacetime is not asymptotically flat. However, we can define ingoing and outgoing waves to be complex linear combinations of the linearly independent solutions at infinity. This leads to the definition [48]

$$R_{\text{in}} = A_{\text{in}} \left(1 - i \frac{cl^2}{r^2} \right), \quad R_{\text{out}} = A_{\text{out}} \left(1 + i \frac{cl^2}{r^2} \right), \quad (68)$$

where c is some positive dimensionless constant, which we take to be independent of the frequency ω , and R_{in} and R_{out} have positive and negative flux, respectively. The ingoing flux is correspondingly

$$\mathcal{F}_{\text{in}} = 8\pi c |A_{\text{in}}|^2. \quad (69)$$

The asymptotic behaviour of (64) for large r is readily available [80], and we can then match this to (68) to determine the coefficients A_{in} and A_{out} . We find

$$\begin{aligned} A_{\text{in}} + A_{\text{out}} &= \frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)}, \\ A_{\text{in}} - A_{\text{out}} &= -\frac{\Delta_- \Gamma(a+b+1)}{icl^2} \left\{ \frac{\log(\Delta_-/l^2) + \psi(a+1) + \psi(b+1) - \psi(1) - \psi(2)}{\Gamma(a)\Gamma(b)} \right. \\ &\quad \left. + \frac{\alpha}{\Gamma(a+1)\Gamma(b+1)} \right\}, \end{aligned} \quad (70)$$

where ψ is the digamma function. We can estimate the relative importance of the two terms in (70) as follows. If $m = 0$ and $\omega \ll \min(\frac{1}{r_+}, \frac{1}{l})$, the difference $A_{\text{in}} - A_{\text{out}}$ in (70) is small compared to the sum $A_{\text{in}} + A_{\text{out}}$, so that

$$A_{\text{in}} \sim \frac{1}{2} \frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)}. \quad (71)$$

This approximation means that the Compton wavelength of the scattered particle is much bigger than the size of the black hole and the scale set by the curvature of the anti-de Sitter space.

Let us consider the $m = 0$ wave and assume $\omega \ll \min(\frac{1}{r_+}, \frac{1}{l})$ so that (71) is valid. Then the partial wave absorption cross section is given by

$$\sigma^{m=0} = \frac{\mathcal{F}(0)}{\mathcal{F}_{\text{in}}} = \frac{1}{\pi c} \mathcal{A}_H \omega \frac{|\Gamma(a+1)\Gamma(b+1)|^2}{|\Gamma(a+b+1)|^2}. \quad (72)$$

In order to relate the partial wave cross section to the plane wave cross section σ_{abs} , we need to divide $\sigma^{m=0}$ by ω [81]. We find

$$\sigma_{\text{abs}} = \mathcal{A}_H \frac{|\Gamma(a+1)\Gamma(b+1)|^2}{|\Gamma(a+b+1)|^2}, \quad (73)$$

where we have chosen c so that $\sigma_{abs}(\omega) \rightarrow \mathcal{A}_H$ for $\omega \rightarrow 0$ [81]. The decay rate Γ of a non-extremal black hole is then given by [48]

$$\begin{aligned}\Gamma &= \frac{\sigma_{abs}}{e^{\frac{\omega}{T_H}} - 1} = T_H \mathcal{A}_H \omega^{-1} e^{-\frac{\omega}{2T_H}} |\Gamma(a+1)\Gamma(b+1)|^2 \\ &= 4\pi^2 l^2 \omega^{-1} T_L T_R e^{-\frac{\omega}{2T_H}} \left| \Gamma\left(1 + i\frac{\omega}{4\pi T_L}\right) \Gamma\left(1 + i\frac{\omega}{4\pi T_R}\right) \right|^2,\end{aligned}\tag{74}$$

where the left and right temperatures are defined by

$$T_{L/R}^{-1} = T_H^{-1} \left(1 \pm \frac{r_-}{r_+}\right),\tag{75}$$

and we have used $\mathcal{A}_H = 2\pi r_+$.

5.3 Microscopic Description

In this subsection, we reverse the usual rules for computing the Hawking decay rate. That is we quantize the black hole while considering the matter field of the last subsection as an external field [49]. Matter fields perturb the dynamics of the metric by acting as sources of energy and momentum. The field Ψ is treated classically, i.e. taken to satisfy the classical wave equation in the bulk of AdS_3 . One may think of this as the curved space equivalent of taking a homogeneous external field in the case of the atom in a radiation field. In this approximation one does not resolve the detailed structure of the bulk. The matter action then reduces to a boundary term

$$S_m = -\frac{1}{2} \int \sqrt{-g} g^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi \rightarrow -\frac{1}{2} \mathcal{B}^r(\infty),\tag{76}$$

where

$$\mathcal{B} = \frac{1}{2} \int_{\partial\mathcal{M}} \sqrt{-g} g^{rr} (\partial_r \Psi^\dagger \Psi + \Psi^\dagger \partial_r \Psi)\tag{77}$$

denotes the boundary term. The operator $\sqrt{-g} g^{rr} \partial_r$ acting on the external field Ψ is the analogue of the dipole operator coupling to an external electric field. How this operator is represented in the microscopic theory of the black hole will, of course, depend on the concrete microscopic model for the black hole (see [49] for details). However, for the present purpose it is enough to note that, according to the results of Brown-Henneaux [29], this operator transforms as a $(1, 1)$ primary field under the asymptotic conformal isometries of AdS_3 . Thus, the external field introduces a perturbation of the CFT at the boundary at infinity by a primary operator $\mathcal{O}(\sigma^+, \sigma^-)$ with conformal weight $(1, 1)$. Here, σ^\pm denote the light cone coordinates on the asymptotic boundary of the black hole metric (1).

Having found the coupling of the external field to the intrinsic degrees of freedom of the black hole, we can now compute transition amplitudes occurring in the presence of a matter field. As explained above, this interaction vertex should correctly describe the transition between black hole states with small energy difference. Note that it is not required that the initial state itself has low energy.

The transition amplitude between an initial and a final state in the presence of an external flux with frequency and angular momentum ω, m is then given by

$$\mathcal{M} = \ell \int d\sigma^+ d\sigma^- \langle f | \mathcal{O}(\sigma^+, \sigma^-) | i \rangle e^{-i(\omega\ell - m)\frac{\sigma^+}{2}} e^{-i(\omega\ell + m)\frac{\sigma^-}{2}}, \quad (78)$$

where i and f denote the initial and final black hole state, respectively. The important point is that calculation of transition amplitudes is reduced to the computation of correlation functions of $(1, 1)$ primary fields.

We proceed to compute the decay rate. For simplicity we set $m = 0$. Squaring the amplitude (78) and summing over final states leads to

$$\sum_f |\mathcal{M}|^2 = \ell^2 \int d\sigma^+ d\sigma^{+'} d\sigma^- d\sigma^{-'} \langle i | \mathcal{O}(\sigma^+, \sigma^-) \mathcal{O}(\sigma^{+'}, \sigma^{-'}) | i \rangle e^{-i\omega\ell\frac{\sigma^+ - \sigma^{+'}}{2}} e^{-i\omega\ell\frac{\sigma^- - \sigma^{-'}}{2}}. \quad (79)$$

Since the black hole corresponds to a thermal state, we must average over initial states weighted by the Boltzmann factor. This means that we must take finite temperature two point functions, which for fields of conformal weight one are given by

$$\langle \mathcal{O}(0, 0) \mathcal{O}(\sigma^+, \sigma^-) \rangle_{T_R, T_L} = \left[\frac{\pi T_R}{\sinh(\pi T_R \sigma^+)} \right]^2 \left[\frac{\pi T_L}{\sinh(\pi T_L \sigma^-)} \right]^2, \quad (80)$$

provided $T \gg V^{-1}$. These have the right periodicity properties in the Euclidean section. The remaining integrals can be performed by contour techniques of common use in thermal field theory. Whether we deal with emission or absorption depends on how the poles at $\sigma^+ = 0$, $\sigma^- = 0$ are dealt with. The choice for emission leads to integrals of the type

$$\int d\sigma^+ \frac{e^{-\frac{i\omega}{2}(\sigma^+ - i\epsilon)}}{\sinh^2(xu)} = \frac{\pi\omega}{x^2} \sum_{n=1}^{\infty} e^{-\frac{\omega\pi n}{2x}} = \frac{\pi\omega}{x^2} \left(e^{\frac{\omega\pi}{2x}} - 1 \right)^{-1}. \quad (81)$$

The resulting emission rate is then given by

$$\Gamma = \frac{\omega\pi^2\ell^2}{(e^{\frac{\omega}{2T_L}} - 1)(e^{\frac{\omega}{2T_R}} - 1)}, \quad (82)$$

where we have included a factor ω^{-1} for the normalization of the outgoing scalar. Eq. (82) reproduces correctly the semiclassical result (74), therefore providing a microscopic derivation of the decay of BTZ black holes. Note that the above analysis is valid for all values of M and J , subject to the low energy restriction. Thus, the conformal field theory description of the BTZ black hole is not restricted to the near-extreme (near BPS) limit. As in the case of the entropy, the relevance of the above calculation to the five-dimensional black hole should also be noted. In particular, the non-trivial part of the greybody factors of the five-dimensional black hole arises due to the presence of the BTZ black hole in the near-horizon limit [82]. Other aspects of the decay rate within the context of the AdS/CFT correspondence are treated in [83]-[86].

6 Outlook

To summarise, we have reviewed various exact results for the BTZ black hole in $2 + 1$ dimensions. In particular, we have shown that this toy model provides the first example of an exact

determination of the Choptuik scaling parameter. We have also seen that the BTZ black hole provides a precise mathematical model of a holographic manifold. Furthermore, we found that the notion of holography (in the sense Sullivan’s theorem) is equivalent to a ‘No Hair Theorem’ for this black hole spacetime. We found that geometrical properties of the spacetime (as encoded is the Brown-Henneaux asymptotic symmetry algebra) are sufficient to yield a microscopic understanding of the Bekenstein-Hawking entropy.

In spite of considerable progress, there are a number of issues that deserve further consideration. For example, one would like to construct spacetimes with the topology of a genus g handlebody [87]-[90]. Indeed, such spacetimes can be constructed via a Schottky procedure, and are known to be geometrically finite [74]. Thus, according to Sullivan’s theorem the resulting spacetimes would be parametrized by $(3g - 3)$ complex Teichmüller parameters. The construction of such spacetimes will play a role in verifying the AdS/CFT correspondence on higher genus Riemann surfaces [90].

An important issue in any discussion of black hole physics is the information puzzle. Given the detailed understanding of various microscopic aspects of the BTZ black hole, one might expect that one should be able to gain a better understanding of the information puzzle in this model. However, a clear resolution of the puzzle remains elusive to date. In particular, it would be useful to understand the precise relation between the counting of microstates presented here and Carlip’s counting of states at the horizon [39].

Acknowledgements

D.B. and S.S. would like to thank Conall Kennedy and Andy Wilkins for collaboration in the results of Ref. [21]. I.S. would like to thank Roberto Emparan for collaboration in the results of Ref. [49]. The work of D.B. and I.S. is part of a project supported by Enterprise Ireland international collaboration grant IC/2000/009.

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